B.Sc Part I (Hons) Paper I

Important Examples

Ex(1) If $x>y$, show that $\tan^{-1}\left(\frac{x+iy}{x-iy}\right)=\frac{\pi}{4}+\frac{i}{2}\log\frac{x+y}{x-y}$

Solution:— Let $x=r\cos\theta$ and $y=r\sin\theta$

then $\tan^{-1}\left(\frac{x+iy}{x-iy}\right)=\tan^{-1}\left(\frac{r\cos\theta+ir\sin\theta}{r\cos\theta-ir\sin\theta}\right)$

$=\tan^{-1}\left(\frac{\cos\theta+i\sin\theta}{\cos\theta-i\sin\theta}\right)$

$=\tan^{-1}\left\{\frac{e^{i\theta}}{e^{-i\theta}}\right\}=\tan^{-1}(e^{i2\theta})$.

Now we require to separate

$\tan^{-1}(e^{i2\theta})=\tan^{-1}(\cos2\theta+i\sin2\theta)$

into real and imaginary parts

let $\tan^{-1}(\cos2\theta+i\sin2\theta)=\alpha+i\beta$ —— (1)

So that $\tan(\alpha+i\beta)=\cos2\theta+i\sin2\theta$

Then $\tan(\alpha-i\beta)=\cos2\theta-i\sin2\theta$

Now $\tan2\alpha=\tan\{(\alpha+i\beta)+(\alpha-i\beta)\}$

$=\frac{\tan(\alpha+i\beta)+\tan(\alpha-i\beta)}{1-\tan(\alpha+i\beta)\tan(\alpha-i\beta)}$

$=\frac{(\cos2\theta+i\sin2\theta)+(\cos2\theta-i\sin2\theta)}{1-(\cos2\theta+i\sin2\theta)(\cos2\theta-i\sin2\theta)}$

$=\frac{2\cos2\theta}{1-(\cos^2 2\theta+\sin^2 2\theta)}=\frac{2\cos2\theta}{1-1}=\frac{2\cos2\theta}{0}$

$=\infty$.

$\Rightarrow 2\alpha=\frac{\pi}{2}\ \therefore\ \alpha=\pi/4$.

Again $\tan2i\beta=\tan\{(\alpha+i\beta)-(\alpha-i\beta)\}$

$=\frac{\tan(\alpha+i\beta)-\tan(\alpha-i\beta)}{1+\tan(\alpha+i\beta)\tan(\alpha-i\beta)}$

$=\frac{(\cos2\theta+i\sin2\theta)-(\cos2\theta-i\sin2\theta)}{1+(\cos2\theta+i\sin2\theta)(\cos2\theta-i\sin2\theta)}$

$=\frac{2i\sin2\theta}{1+(\cos^2 2\theta+\sin^2 2\theta)}=\frac{2i\sin2\theta}{1+1}=i\sin2\theta$

$$\Rightarrow i\tanh 2\beta = i\sin 2\theta \Rightarrow \tanh 2\beta = \sin 2\theta$$

$$\Rightarrow 2\beta = \tanh^{-1}\sin 2\theta = \frac{1}{2}\log\frac{1+\sin 2\theta}{1-\sin 2\theta}$$

$$= \frac{1}{2}\log\frac{(\cos\theta+\sin\theta)^2}{(\cos\theta-\sin\theta)^2} = \log\frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}$$

$$= \log\frac{\frac{x}{r}+\frac{y}{r}}{\frac{x}{r}-\frac{y}{r}} \quad \text{Since } x = r\cos\theta \text{ and } y = r\sin\theta$$

$$= \log\frac{x+y}{x-y} \quad \therefore \beta = \frac{1}{2}\log\frac{(x+y)}{(x-y)}$$

Hence from (1), $\tan^{-1}(\cos 2\theta + i\sin 2\theta) = \alpha + i\beta$

i.e. $\tan^{-1}\dfrac{x+iy}{x-iy} = \dfrac{\pi}{4} + \dfrac{i}{2}\log\dfrac{x+y}{x-y}$

Ex:— If $\sin^{-1}(x+iy) = \tan^{-1}(u+iv)$, show that

$$[(x-1)^2+y^2][(x+1)^2+y^2] = \left(\frac{x^2+y^2}{u^2+v^2}\right)^2$$

Solution:— We know that

$$\sin^{-1}\theta = \tan^{-1}\frac{\theta}{\sqrt{1-\theta^2}}$$

$$\therefore \sin^{-1}(x+iy) = \tan^{-1}\frac{x+iy}{\sqrt{\{1-(x+iy)^2\}}}$$

$$= \tan^{-1}(u+iv), \text{ given.}$$

$$\therefore \frac{x+iy}{\sqrt{\{1-(x+iy)^2\}}} = u+iv \quad \text{———— (1)}$$

Writing the corresponding conjugate, we have

$$\frac{x-iy}{\sqrt{\{1-(x-iy)^2\}}} = u-iv \quad \text{———— (2)}$$

Multiplying (1) and (2), we get

$$\frac{x^2+y^2}{\sqrt{\{(1-x^2+y^2)^2+4x^2y^2}} = u^2+v^2$$

$$\Rightarrow \frac{x^2+y^2}{u^2+v^2} = \sqrt{\{\{(1-x^2+y^2)^2+4x^2y^2\}} \quad \text{———— (3)}$$

Now the R.H.S $= (1-x^2+y^2)^2 + 4x^2y^2$

$$= (1+x^2+y^2)^2 - 4x^2(1+y^2) + 4x^2y^2$$
$$= (1+x^2+y^2)^2 - 4x^2$$
$$= (1+x^2+2x+y^2)(1+x^2-2x+y^2)$$
$$= [(x+1)^2+y^2][(x-1)^2+y^2]$$

Hence from (3) the result follows

Ex:— Prove that $\tan^{-1}\left[i\,\frac{x-a}{x+a}\right] = -\frac{i}{2}\log a/x$

Solution:— $L.H.S = \tan^{-1}\left[i\,\frac{x-a}{x+a}\right] = i\tanh^{-1}\left[\frac{x-a}{x+a}\right]$

Since $\tanh^{-1}w = -i\tan^{-1}(iw)$

But $\tanh^{-1}w = \frac{1}{2}\log\frac{1+w}{1-w}$

$$\therefore L.H.S = i\cdot\frac{1}{2}\log\, 1+\frac{x-a}{x+a} \Big/ 1-\frac{x-a}{x+a}$$

$$= \frac{i}{2}\log\frac{(x+a)+(x-a)}{(x+a)-(x-a)} = \frac{i}{2}\log\frac{2x}{2a}$$

$$= \frac{i}{2}\log x/a = -\frac{i}{2}\log a/x = R.H.S.$$

$L.H.S = R.H.S$

Proved

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